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A two-component generalization of the Degasperis–Procesi equation

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Abstract

We present two different Hamiltonian extensions of the Degasperis–Procesi equation to two-component equations. The construction is based on the observation that the second Hamiltonian operator of the Degasperis–Procesi equation could be considered as the Dirac-reduced Poisson tensor of the second Hamiltonian operator of the Boussinesq equation. The first extension is generated by the Hamiltonian operator which is a Dirac-reduced operator of the generalized but degenerated second Hamiltonian operator of the Boussinesq equation. The second one is obtained by the $N = 2$ supersymmetric extension of the aforementioned method. As the byproduct of this procedure, we obtain the Hamiltonian system of interacting equations which contains the Camassa–Holm and Degasperis–Procesi equations.

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1. Introduction

Recently, a family of equations of the form [1–7]

$$u_t - u_{xxt} = \frac{1}{2}\left(-(b+1)u^2 + 2uu_{xx} + (b-1)u_x^2\right)_x \tag{1}$$

has been investigated in the literature.

When $b = 2$, equation (1) reduces to the Camassa–Holm equation

$$u_t - u_{xxt} = \frac{1}{2}\left(-3u^2 + 2uu_{xx} + u_x^2\right)_x, \tag{2}$$

which describes a special approximation of shallow water theory. This equation shares most of the important properties of an integrable system of a KdV type, for example, the existence of Lax pair formalism, the bi-Hamiltonian structure and the multi-soliton solutions. Moreover, this equation admits peaked solitary wave solutions.

Degasperis and Procesi showed that equation (1) is also integrable for the $b = 3$ case. The Degasperis–Procesi equation

$$u_t - u_{xxt} = \left(-2u^2 + uu_{xx} + u_x^2\right)_x \tag{3}$$

can also be considered as a model for shallow water dynamics and found to be completely integrable. Similar to the Camassa–Holm case, the Degasperis–Procesi equation has the Lax pair and also admits peakon dynamics.

The Camassa–Holm and Degasperis–Procesi equations possess the bi-Hamiltonian structure and recursion operators. In the case of the Camassa–Holm equation, there are two local Hamiltonian structures, given by

$$\begin{aligned} &m_t = B_o \frac{\delta H_2}{\delta m} = B_1 \frac{\delta H_1}{\delta m} \qquad B_o = -\partial(1-\partial^2) = -\mathcal{L}, \qquad B_1 = -(m\partial + \partial m) \\ &H_2 = \frac{1}{2}\int \mathrm{d}x\left(u^3 + uu_x^2\right), \qquad H_1 = \frac{1}{2}\int \mathrm{d}x\left(u^2 + u_x^2\right) \end{aligned} \tag{4}$$

with $m = u - u_{xx}$, whose compatibility was known in [5]. In the case of the Degasperis–Procesi equation, there is only one local Hamiltonian structure and the second Hamiltonian structure is nonlocal:

$$\begin{aligned} &m_t = B_o \frac{\delta H_{-1}}{\delta m} = B_1 \frac{\delta H_o}{\delta m} \qquad B_o = \mathcal{L}(4-\partial^2), \\ &B_1 = (m_x + 3m\partial)\mathcal{L}^{-1}(2m_x + 3m\partial) \qquad H_{-1} = -\frac{1}{6}\int u^3\,\mathrm{d}x, \qquad H_o = -\frac{1}{2}\int m\,\mathrm{d}x, \end{aligned} \tag{5}$$

whose compatibility was proven in [6].

The two-component generalization of the Camassa–Holm equation

$$m_t = -um_x - 2mu_x + \rho\rho_x \qquad \rho_t = -(\rho u)_x, \tag{6}$$

where $m = u - u_{xx}$, has been recently proposed by Falqui [8] and Chen et al [9]. This generalization, similar to the Camassa–Holm equation, is the first negative flow of the AKNS hierarchy and possesses the peakon and multi-kink solutions and the bi-Hamiltonian structure [9, 10, 8]. The basic idea of this generalization was to include the additional function to the Lax pair and then to extract the basic properties of the equation from this generalized Lax pair representation.

In this paper, we show that it is possible to construct two different generalizations of the two-component version of Degasperis–Procesi equations as Hamiltonian equations in the form

$$\rho_t = -k_2\rho_x u - (k_1 + k_2)\rho u_x \qquad m_t = -3mu_x - m_x u + k_3\rho\rho_x, \tag{7}$$

where $m = \partial^{-1}\mathcal{L}u$ while $k_1 = k_2 = 1$ and k_3 is an arbitrary constant or $k_2 = 1, k_3 = 0$ and k_1 takes an arbitrary value.

The second generalization is

$$\rho_t = -2\rho u_x - \rho_x u \qquad m_t = -3mu_x - m_x u - \rho u_x + 2\rho\rho_x, \tag{8}$$

where $m = \partial^{-1}\mathcal{L}u$.

We also show that it is possible to construct the interacting system of equations which contains the Camassa–Holm and Degasperis–Procesi equations:

$$m_t = -3m(2u_x + v_x) - m_x(2u + v) \qquad n_t = -2n(2u_x + v_x) - n_x(2u + v), \tag{9}$$

where $m = u - u_{xx}, n = v - v_{xx}$.

The construction presented in this paper is based on the generalization of the second Hamiltonian operator B_1 of the Degasperis–Procesi equation to the two-dimensional matrix operator. The direct manner of the generalization leads us to very complicated assumptions on the entries of the matrix and very difficult verifications of the Jacobi identity. We omit these complications in three steps. In the first step, which we will call the decompression of the Hamiltonian operators, we consider the Hamiltonian pencil $B_o + B_1$ of the Degasperis–Procesi equation, as the Dirac-reduced operator of the second Hamiltonian operator of the Boussinesq equation. As a result, we obtained the two-dimensional local matrix Hamiltonian operator. In the second step, we generalize this operator to the three-dimensional matrix operator such that the Jacobi identity is fulfilled. In the last step, we apply the Dirac reduction with respect to the decompression function appearing in the first step.

The paper is organized as follows. In the first section, we describe the Dirac reduction technique and we show that the Hamiltonian pencil of the Camassa–Holm equation follows from the second Hamiltonian operator of the nonlinear Schrödinger equation. In the second section, we decompress the Hamiltonian operator of the Degasperis–Procesi equation to the second Hamiltonian operator of the Boussinesq equation. In the third section, we carry out the Dirac reduction of the generalized, but degenerated Hamiltonian operator of the Boussinesq equation. The first two-component generalization of the Degasperis–Procesi equation is presented in the third section. The fourth section contains the description of the interacting system of Camassa–Holm and Degasperis–Procesi equations. The fifth section contains the supersymmetric investigation of the decompression method which allowed us to obtain the second generalization of the two-component Degasperis–Procesi equation. The last section contains concluding remarks.

2. The Dirac reduction of the Poisson tensor

The energy-dependent Schrödinger spectral problem [11] for the Camassa–Holm equation can be formulated with the help of the Lax operator as

$$\Psi_{xx} = \left(\frac{1}{4} - \lambda m\right)\Psi \qquad \Psi_t = -\left(\frac{1}{2\lambda} + u\right)\Psi_x + \frac{1}{2}u_x\Psi. \tag{10}$$

The compatibility condition for the above system yields two independent equations:

$$m_t = -2mu_x - m_x u \qquad m = u - u_{xx}. \tag{11}$$

We would like to obtain the Hamiltonian operator for the Camassa–Holm equation and therefore we consider a more general system than (10):

$$\Psi_{xx} = v\Psi_x - w\Psi \qquad \Psi_t = -A_1\Psi_x + A_2\Psi, \tag{12}$$

where now w, v are given functions while A_1, A_2 are at the moment arbitrary functions. This system can be reduced to the Lax representation (10) for the special choice of the functions w, v, A_1, A_2 and if we additionally assume the dependence on the spectral parameter. The compatibility conditions for equation (12) give us the following time evolution of the functions v, w:

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = J\begin{pmatrix} A_2 \\ -A_1 \end{pmatrix} = \begin{pmatrix} 2\partial & \partial v + \partial^2 \\ -\partial^2 + v\partial & \partial w + w\partial \end{pmatrix}\begin{pmatrix} A_2 \\ -A_1 \end{pmatrix}. \tag{13}$$

To establish the Hamiltonian character of the corresponding flows (13), we have to choose A_1 and A_2 in such a way that they constitute the coordinates of variational derivatives of some functionals $\mathcal{H}$. However, we follow in a different way. Let us note that our J operator is the second Hamiltonian operator connected with the AKNS equations. Indeed under the 'coordinate change'

$$v = \frac{q_x}{q}, \qquad w = pq, \tag{14}$$

the J operator transforms to

$$J = \begin{pmatrix} -2p\partial^{-1}p & \partial + 2p\partial^{-1}q \\ \partial + 2q\partial^{-1}p & -2q\partial^{-1}q \end{pmatrix}. \tag{15}$$

A perhaps less-known fact is the following. Under the Dirac reduction where $q = 1$ or $p = 1$, this Hamiltonian reduces to the second Hamiltonian operator for the Korteweg–de Vries equation. We use the standard reduction lemma for Poisson brackets [12] which can be stressed as, for the given Poisson tensor,

$$P(v, w) = \begin{pmatrix} P_{vv}(v, w) & P_{vw}(v, w) \\ P_{wv}(v, w) & P_{ww}(v, w) \end{pmatrix}. \tag{16}$$

Let us assume that $P_{vv}(v, w)$ is invertible, then for arbitrary v the map given by

$$\Theta(w; v) = P_{ww}(v, w) - P_{wv}(v, w)(P_{vv}(v, w))^{-1}P_{vw}(v, w) \tag{17}$$

is a Poisson tensor where v enters the reduced Poisson tensor Θ as a parameter rather than as a variable. The reduced Poisson tensor $\Theta(v : w)$ reads

$$\Theta(v; w) = P_{vv}(v, w) - P_{vw}(v, w)(P_{ww}(v, v))^{-1}P_{wv}(v, w). \tag{18}$$

Now we can apply this reduction to the Hamiltonian operator defined in equation (13) where we assume that $v = 1$ and as a result we obtain the following operator:

$$\Theta = -\tfrac{1}{2}(\partial - \partial^3) + \partial w + w\partial. \tag{19}$$

It appears that this operator is the linear combination of our first and second Hamiltonian operators of the Camassa–Holm equation.

On the other hand, we can carry out the Dirac reduction with respect to the function w where now $w = 1$. As a result, we obtained the following Poisson tensor:

$$\Theta = 2\partial - \tfrac{1}{2}\partial^3 - \tfrac{1}{2}\partial v\partial^{-1}v\partial, \tag{20}$$

which is the linear combination of the first and second Hamiltonian operators of the modified Korteweg–de Vries equation.

3. The decompression of the Hamiltonian pencil of the Degasperis–Procesi equation

The energy-dependent Lax operator responsible for the Degasperis–Procesi equation is [3, 4]

$$\Psi_{xxx} = \Psi_x - \lambda m\Psi \qquad \Psi_t = -\lambda^{-1}\Psi_{xx} - u\Psi_x + u_x\Psi, \tag{21}$$

where λ is the parameter. The compatibility conditions give us the Degasperis–Procesi equation.

In order to obtain the Hamiltonian operator for the Degasperis–Procesi equation, we consider a more general system of equations than (21):

$$\begin{aligned} \Psi_{xxx} &= v\Psi_x + \tfrac{1}{2}(v_x + 2z)\Psi \\ \Psi_t &= A\Psi_{xx} + \left(A_1 - \tfrac{1}{2}A_x\right)\Psi_x + \tfrac{1}{6}(A_{xx} - 6A_{1,x} - 4vA)\Psi, \end{aligned} \tag{22}$$

where v, z are given functions while A, A_1 are at the moment arbitrary functions. This system can be reduced to the Lax representation (21) for the special choice of the functions v, z, A, A_1 and if we additionally assume the dependence on the spectral parameter.

The compatibility conditions for equation (22) give us

$$\begin{pmatrix} v \\ z \end{pmatrix}_t = J\begin{pmatrix} A_1 \\ A \end{pmatrix} = \begin{pmatrix} -2\partial^3 + 2v\partial + v_x & 3z\partial + 2z_x \\ 3z\partial + z_x & \frac{1}{12}J_{2,2} \end{pmatrix}\begin{pmatrix} A_1 \\ A \end{pmatrix}, \tag{23}$$

where

$$J_{2,2} = 2\partial^5 - 10v\partial^3 - 15v_x\partial^2 + (8v^2 - 9v_{xx})\partial + 8v_x v - 2v_{xxx}. \tag{24}$$

We recognize that J is the second Hamiltonian operator for the Boussinesq equation. We can easily obtain the Boussinesq equation assuming $A_1 = 0$, $A = 1$ which gives us

$$v_t = 2z_x \qquad z_t = (-2v_{xxx} + 8v_x v)/12. \tag{25}$$

Let us now investigate the behaviour of the J operator under the Dirac reduction where we assume $v = 1$. Using formula (17), we obtained the following Poisson tensor:

$$B = \tfrac{1}{6}\partial(4 - \partial^2)(1 - \partial^2) - \tfrac{1}{2}(3z\partial + z_x)(\partial - \partial^3)^{-1}(3z\partial + 2z_x). \tag{26}$$

We quickly recognize, after the identification $z = m$, that B is the linear combination of the first and second Hamiltonian operators of the bi-Hamiltonian structure of the Degasperis–Procesi equation. Thus, the B operator

$$B = \tfrac{1}{6}B_o - \tfrac{1}{2}B_1 \tag{27}$$

satisfies the Jacobi identity due to the Dirac reduction. Moreover using the scaling argument to the function z, we can easily verify that B_o and B_1 are the compatible operators and that B_1 satisfy the Jacobi identity as well.

We call this process as the decompression of the Hamiltonian structure. More precisely in this procedure we try to find a higher dimensional operator for which the Dirac reduction gives us the Hamiltonian operator under consideration. In some sense, it is an inverse operation to the Dirac reduction technology. The advantage of this decompression technique is a possibility of quick verification of the Jacobi identity for the nonlocal Hamiltonian operators. Indeed if we embed some nonlocal Hamiltonian operators in such a way that the final operator will be local, then it is much easier to check the Jacobi identity compare to the verification of this identity for the nonlocal operators. For example, the decompressed second Hamiltonian operator of the AKNS equations (15) gives us the local Hamiltonian operator which is connected with the Kac–Moody $sl(2)$ algebra [13]. The disadvantage of this technique is a lack of uniqueness because we can also embed the given Hamiltonian operator to the higher dimensional matrix operator in a different manner, as we will see in the following section.

It is hard to define the general prescription of the decompression procedure using the examples mentioned earlier. In the case of the nonlocal Hamiltonian operators, this construction requires many assumptions if we would like to obtain, as the final result, the higher dimensional local operator. However, in this paper we would like to extend the J operator defined in equation (23) to the three-dimensional Hamiltonian matrix operator including new function such that the gradation of the matrix elements with respect to the weights of the functions is preserved. In the next step we carry out the Dirac reduction for this extended matrix operator when $v = 1$ and obtain some Hamiltonian operator. Then we use this new Hamiltonian operator to the construction of the two-component Degasperis–Procesi equations.

To finish this section, let us note that it is possible to consider a more general form of the Hamiltonian operator than that defined by equation (23). Indeed, let us consider the following operator:

$$\mathcal{J} = \begin{pmatrix} c\partial^3 + 2v\partial + v_x & 3z\partial + 2z_x \\ 3z\partial + z_x & \mathcal{J}_{2,2} \end{pmatrix}, \tag{28}$$

where c is an arbitrary central extension term and $\mathcal{J}_{2,2}$ is constructed out of the function v, its derivatives and from the differential operators only. The verification of the Jacobi identity leads us to the conclusion that this identity holds if $\mathcal{J}_{2,2}$ is defined as

$$\mathcal{J}_{2,2} = \frac{\lambda}{16}(c^2\partial^5 + 10cv\partial^3 + 15cv_x\partial^2 + (9cv_{xx} + 16v^2)\partial + 2cv_{xxx} + 16v_x v), \tag{29}$$

where λ is an arbitrary constant. We see that $\mathcal{J}$ reduces to the J operator when $c = -2$ and $\lambda = \frac{2}{3}$. The most interesting case is the degenerated one where we assume that $\lambda = 0$ while c is an arbitrary constant. For this degenerated case, it is also possible to construct the second Hamiltonian operator for the Degasperis–Procesi equation using the decompression described earlier. However, the information on the first Hamiltonian operator is then lost.

4. A two-component Degasperis–Procesi equation

Let us decompress the $\mathcal{J}$ operator defined by equation (28) to the three-dimensional matrix Γ including the new function ρ which has the same weight as the v function. We try to find the general form of this operator such that the gradation of the matrix elements with respect to the weights of the functions is preserved.

Let us first consider the decompression for the degenerated case where $\lambda = 0, c = -2$ and $z = m$. We make the following assumptions on the entries of the Γ matrix $\Gamma_{1,1} = J_{1,1}, \Gamma_{1,2} = J_{1,2}, \Gamma_{2,1} = J_{2,1}$ where $J_{1,1}, J_{1,2}, J_{2,1}$ are defined in equation (23). For the rest of the elements, we assumed that they are constructed out of the function ρ, its derivatives and differential operators in such a way that they are reduced to 0 when $\rho = 0$. We checked, using computer algebra, that the following matrix:

$$\Gamma = \begin{pmatrix} -2\partial^3 + 2v\partial + v_x & 3m\partial + 2m_x & k_1\partial\rho + k_2\rho\partial \\ 3m\partial + m_x & \frac{1}{2}k_3\rho\partial\rho & 0 \\ k_2\partial\rho + k_1\rho\partial & 0 & 0 \end{pmatrix} \tag{30}$$

satisfies the Jacobi identity for the two choices of free parameters: $k_1 = k_2 = 1$ and k_3 is an arbitrary value for the first case while $k_2 = 1, k_3 = 0$ and k_1 is an arbitrary value for the second case.

If we, similar to the previous case, carry out the Dirac reduction with respect to $v = 1$, we obtain the following matrix Hamiltonian operator:

$$\mathcal{Z} = -\frac{1}{2}\begin{pmatrix} \begin{matrix}(3m\partial + m_x)\mathcal{L}^{-1}(3m\partial + 2m_x) \\ -k_3\rho\partial\rho\end{matrix} & (3m\partial + m_x)\mathcal{L}^{-1}(k_1\partial\rho + k_2\rho\partial) \\ (k_2\partial\rho + k_1\rho\partial)\mathcal{L}^{-1}(3m\partial + 2m_x) & (k_2\partial\rho + k_1\rho)\mathcal{L}^{-1}(k_1\partial\rho + k_2\rho\partial) \end{pmatrix}. \tag{31}$$

Let us compute the equation of motion for the Hamiltonian $H = \int \mathrm{d}x m$. Assuming that $m = u - u_{xx}$, we obtain

$$m_t = k_3\rho\rho_x/2 - 3mu_x - m_x u \qquad \rho_t = -k_2\rho_x u - (k_1 + k_2)\rho u_x. \tag{32}$$

It is our two-component generalization of the Degasperis–Procesi equation. In the case $k_3 = 0$, the two equations (32) are no more coupled and the equation on ρ becomes linear.

We tried to find the first Hamiltonian operator for the system (34), decompressing the $\mathcal{J}$ operator defined in equation (28) in the nondegenerated case. Unfortunately, we have not been able to find any such operator and moreover we did not define any new generalization of the Degasperis–Procesi equation in that manner. We make the same assumptions on the decompressed matrix $\hat{\Gamma}$ as earlier with the restriction. The element $\hat{\Gamma}_{2,2}$ is constructed out of the functions v, ρ, its derivatives and differential operators and it reduces to $\mathcal{J}_{2,2}$ when $\rho = 0$. We fixed the $\hat{\Gamma}$ matrix verifying the Jacobi identity, carried out the Dirac reduction of the $\hat{\Gamma}$ matrix with respect to the function $v = 1$ and obtained the reduced matrix in the form, $\Omega = \mathcal{Z}_o + \mathcal{Z}$, where $\mathcal{Z}$ is defined by equation (31). The computer algebra identified only one operator $\mathcal{Z}_o$ which, however, as we checked does not satisfy the Jacobi identity. It means that $\mathcal{Z}_o$ is not the first Hamiltonian operator for the system (32).

We have been not able to find any additional constants of motion for the system (32). We tried to find these constants from the Lax representation. Therefore, we verified two different assumptions on the Lax operator which should give us the two-component Degasperis–Procesi equations. The first was the matrix generalization of the Lax operator responsible for the Degasperis–Procesi equation while in the second we assumed the polynomial dependence of the spectral parameter.

Unfortunately, we did not find any Lax representation for the two-component Degasperis–Procesi equations and hence did not establish the integrability of the system in that manner. However, it does not mean that this system is not integrable. We need quite different methods in order to establish the integrability of the system (34) as, for example, to try to find the recursion operator. It seems that the problem of the existence of higher order constants of motion and the recursion operator for the system (34) is worth studying.

5. Degasperis–Procesi equation interacted with Camassa–Holm equation

Let us consider the case when $k_3 = 0$ and $k_1 = k_2 = 1$ and redefine the variables as $\rho = n = v - v_{xx}$; then the Hamiltonian operator $2\mathcal{Z}$ defines a new Hamiltonian operator

$$Z = -\begin{pmatrix} 9m^{2/3}\partial m^{1/3}\mathcal{L}^{-1}m^{1/3}\partial m^{2/3} & 6m^{2/3}\partial m^{1/3}\mathcal{L}^{-1}n^{1/2}\partial n^{1/2} \\ 6n^{1/2}\partial n^{1/2}\mathcal{L}^{-1}m^{1/3}\partial m^{2/3} & 4n^{1/2}\partial n^{1/2}\mathcal{L}^{-1}n^{1/2}\partial n^{1/2} \end{pmatrix}, \tag{33}$$

where $\mathcal{L}^{-1} = \partial^{-1}(1-\partial^2)^{-1}$. This operator satisfies the Jacobi identity due to the Dirac reduction of the Γ operator defined by equation (30). Thus, one can define the following equations of motion:

$$\begin{aligned} m_t &= -3m(2u_x + v_x) - m_x(2u + v) \\ n_t &= -2n(2u_x + v_x) - n_x(2u + v), \end{aligned} \tag{34}$$

if we apply the Z operator to the Hamiltonian $H = \int \mathrm{d}x\,(m + n)$. It is our interacting system of equations which contains the Camassa–Holm and Degasperis–Procesi equations. Indeed when $n = v = 0$ and we rescale the time our system reduces to the Degasperis–Procesi equation while the reduction $m = u = 0$ leads us to the Camassa–Holm equation.

To our best knowledge, it is a new system of equations and one can ask whether this system is integrable. We have found three independent conserved quantities

$$\begin{aligned} H_0 &= \int (m+n)\,\mathrm{d}x \qquad H_1 = \int n^{\lambda} m^{(1-2\lambda)/3}\,\mathrm{d}x \\ H_2 &= \int \left(-9n_x^2 n^{\lambda-2} m^{-(1+2\lambda)/3} + 12 n_x m_x n^{\lambda-1} m^{-(4+2\lambda)/3} - 4m_x^2 n^{\lambda} m^{-(7+2\lambda)/3}\right)\mathrm{d}x, \end{aligned} \tag{35}$$

where λ is an arbitrary constant. The H_1 conserved quantity is the Casimir function for our Hamiltonian operator (33). Interestingly when $\lambda = 0$ then H_1 reduces to the Casimir function for the Degasperis–Procesi equation, while for $\lambda = 1/2$ it reduces to the Casimir function for the Camassa–Holm equation. The existence of three independent conserved quantities is a good sign to expect that this system is integrable.

The popular manner of checking the integrability is to define the recursion operator using the bi-Hamiltonian formulation. However, we could not find such a structure. On the other hand, the easiest manner of verifying the integrability is to define the Lax representation for the given partial differential equation. If such a representation exists for our system (34), then this should be reduced to the Degasperis–Procesi or to the Camasaa-Holm Lax representation when $n = v = 0$ or $m = u = 0$, respectively. One can, therefore, think that the system of interacting Camassa–Holm and Degasperis–Procesi equations appears as the multi-component generalization of the Lax operator responsible for the Degasperis–Procesi equation. It is well known that an extension of a scalar integrable partial differential equation to a multi-component version, as for example for the vector nonlinear Schrödinger equation, is still integrable and can be achieved by considering the corresponding Lax pair in a higher rank matrix algebra. We verified such a possibility and therefore considered the most general assumption on the two-dimensional matrix generalization of the Lax operator of the Degasperis–Procesi equation which also contained the Lax operator of the Camasaa–Holm equation. However, we did not find any operator which produces the system (34). The difficulties in such a construction are probably connected with the different orders of the differential operators in the Lax operators of the Camassa–Holm and Degasperis–Procesi equations.

The next possibility of checking the integrability is to consider the third-order energy-dependent scalar Lax operator where the polynomial dependence of the spectral parameter is assumed [14]. It was shown in [9, 10] that if one allows the polynomial dependence of the scalar parameter for the second-order energy-dependent scalar Lax operator, then this leads us to the two-component generalization of the Camassa–Holm equation. We have checked that the same strategy cannot be applied for the Degarsperis–Procesi equation.

6. The extended $N = 2$ supersymmetric Degasperis–Procesi equation

We will use now the supersymmetric formalism [15] which allows us to consider the supersymmetric analogue of the second Hamiltonian operator which is connected with the degenerated second Hamiltonian operator of the Boussinesq equation. Here, we will use the supersymmetric algebra of (super) derivatives where

$$\mathcal{D}_1 = \frac{\partial}{\partial\theta_1} - \frac{1}{2}\theta_2\frac{\partial}{\partial x} \qquad \mathcal{D}_2 = \frac{\partial}{\partial\theta_2} - \frac{1}{2}\theta_1\frac{\partial}{\partial x}$$
$$\{\mathcal{D}_1, \mathcal{D}_2\} = -\partial, \qquad [\mathcal{D}_1, \mathcal{D}_2] = \mathcal{D}_1\mathcal{D}_2 - \mathcal{D}_2, \mathcal{D}_1, \qquad \mathcal{D}_1^2 = \mathcal{D}_2^2 = 0. \tag{36}$$

The superfunctions can be thought as the $N = 2$ supermultiplets which depend on x and, additionally, on two Grassman-valued functions with the entries

$$u(x, \theta_1, \theta_2) = u_o(x) + \theta_1\chi_1(x) + \theta_2\chi_2(x) + \theta_2\theta_1 u_1(x) \tag{37}$$

where $u_o(x)$, $u_1(x)$ are the classical functions, $\chi_1(x)$, $\chi_2(x)$ are Grassmann-valued functions and θ_1, θ_2 are the Majorana spinors [15].

The main idea of the supersymmetry is to treat boson and fermion operators equally. In order to get a supersymmetric theory, we have to add to a system of k bosonic equations kN fermions and $k(N-1)$ boson fields $k = 1, 2, \ldots, N$, $N = 1, 2, \ldots$, in such a way that the final theory becomes supersymmetric invariant. From the soliton point of view, we can distinguish two important classes of supersymmetric equations: the non-extended ($N = 1$) and extended ($N > 1$) cases. Consideration of the extended case may imply new bosonic equations whose properties need further investigation.

There are many different methods of the supersymmetrization of the classical equations, and many new integrable equations [16–19] have been discovered in that manner. For example, Devchand and Schiff [20] recently found non-extended supersymmetric generalization of the Camassa–Holm equation. The present author showed [21] that extended $N = 2$ supersymmetric generalization of the Camassa–Holm leads us directly to the two-component generalization of this equation considered in [8, 9].

The $N = 2$ supersymmetric Boussinesq equation has been constructed utilizing the $N = 2$ supersymmetric extension of the W_3 algebra [22, 23]. This supersymmetric algebra is generated by two $N = 2$ supermultiplets, with the conformal spins $\left(1, \frac{3}{2}, \frac{3}{2}, 2\right)$ and $\left(2, \frac{5}{2}, \frac{5}{2}, 3\right)$

and exists at an arbitrary value of the central charge and is connected with the following supersymmetric matrix operator $\hat{J}$ with the entries

$$\begin{aligned}
\hat{J}_{1,1} &= c[\mathcal{D}_1, \mathcal{D}_2]\partial + u_x + u\partial + (\mathcal{D}_1 u)\mathcal{D}_2 + (\mathcal{D}_2 u)\mathcal{D}_1 \\
\hat{J}_{1,2} &= 2\partial v + (\mathcal{D}_1 v)\mathcal{D}_2 + (\mathcal{D}_2 v)\mathcal{D}_1 \\
\hat{J}_{2,1} &= \partial v + v\partial + (\mathcal{D}_1 v)\mathcal{D}_2 + (\mathcal{D}_2 v)\mathcal{D}_1,
\end{aligned} \tag{38}$$

where c is an arbitrary constant and the element $\hat{J}_{2,2}$ has a rather complicated form [22, 24]. For the next purposes, we assume that $c = -1$.

However, we do not use this operator for our considerations because if we carry out the Dirac reduction with respect to $u = 1$, it appears that $\hat{J}_{2,2}$ does not satisfy the Jacobi identity. On the other hand, the supersymmetric extension of the W_3 algebra is unique, when $\hat{J}_{2,2} \neq 0$, so we restrict the consideration to the degenerated case where

$$\hat{J}_{2,2} = 0. \tag{39}$$

The $\hat{J}$ matrix operator given by previous equations defines a proper Hamiltonian operator that can be easily checked computing the Jacobi identity. We can apply the Dirac reduction scheme in the supersymmetric case as well. Let us carry out this reduction where $u = 1$, obtaining

$$\Theta = -(\partial v + v\partial + (\mathcal{D}_1 v)\mathcal{D}_2 + (\mathcal{D}_2 v)\mathcal{D}_1)\hat{\mathcal{L}}(2\partial v + (\mathcal{D}_1 v)\mathcal{D}_2 + (\mathcal{D}_2 v)\mathcal{D}_1), \tag{40}$$

where

$$\hat{\mathcal{L}} = \left(\partial - [\mathcal{D}_1, \mathcal{D}_2]\right)^{-1} = \partial^{-1}(1 - \partial^2)^{-1}(1 + [\mathcal{D}_1, \mathcal{D}_2]) = \mathcal{L}^{-1}(1 + [\mathcal{D}_1, \mathcal{D}_2]). \tag{41}$$

This reduced operator generates the following equation of motion when it acts on the Hamiltonian $H = \frac{1}{2}\int \mathrm{d}x\, \mathrm{d}\theta_1\, \mathrm{d}\theta_2 v$:

$$v_t = -(2vA_x + v_x A + (\mathcal{D}_1 v)(\mathcal{D}_2 A) + (\mathcal{D}_2 v)(\mathcal{D}_1 A)), \tag{42}$$

where $A = \partial\hat{\mathcal{L}}v$.

It is our supersymmetric extension of the Degasperis–Procesi equation. We have not been able to find any supersymmetric Lax representation responsible for this supersymmetric equation.

Let us compute the bosonic sector, where all fermionic components disappear; it means that we consider the superfunctions in the form

$$A = A_o + \theta_2\theta_1 A_1 \qquad v = V_o + \theta_2\theta_1 V_1 = (A_o - 2A_1) + \theta_2\theta_1(A_1 - A_{o,xx}/2). \tag{43}$$

In these coordinates, we have

$$V_{o,t} = -2V_o A_{o,x} - V_{o,x} A_o \qquad V_{1,t} = -3V_1 A_{o,x} - V_{1,x} A_o - 2V_o A_{1,x}. \tag{44}$$

In order to have the connection with the Degasperis–Procesi equation, let us introduce new variables ρ and u

$$V_o = \rho, \qquad V_1 = \tfrac{1}{2}(m - \rho), \qquad m = u - u_{xx}, \tag{45}$$

in which

$$A_0 = u, \qquad A_1 = \tfrac{1}{2}(u - \rho). \tag{46}$$

Then, equation (44) transforms to

$$\rho_t = -2\rho u_x - \rho_x u \qquad m_t = -3mu_x - m_x u - \rho u_x + 2\rho\rho_x. \tag{47}$$

In that manner, we obtained the second two-component generalization of the Degasperis–Procesi equation.

7. Conclusion

In this paper we considered two different extensions of the Degasperis–Procesi equation. Our construction is based on the observation that the second Hamiltonian operator of the Degasperis–Procesi equation could be considered as the Dirac-reduced Poisson tensor of the second Hamiltonian operator of the Boussinesq equation. The first extension is generated by the Hamiltonian operator which is obtained as a Dirac-reduced operator of the generalized but degenerated second Hamiltonian operator of the Boussinesq equation. The second one is generated by the supersymmetric $N = 2$ extension. Unfortunately, we did not find any Lax representation for the two-component Degasperis–Procesi equations and hence we have not been able to verify the integrability of the systems. We also presented the interacting system of equations which contains the Camassa–Holm and Degasperis–Procesi equations. For this interacting system we constructed few conserved quantities. The decompression method presented here does not allow us to construct the first Hamiltonian operators for our systems because this method is based on the extensions of some local Hamiltonian operators which eliminate this structure from the very beginning. However, it does not mean that this structure does not exist. If the first Hamiltonian structures appear in our systems, then we need quite different methods in order to find these. It seems that the problem of the existence of the recursion operator and the integrability of our systems is very tempting to study.

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